

# Harmonic maps, Bäcklund-Darboux transformations and "line soliton" analogs

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**Short title.** Harmonic Maps and GBDT

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## Abstract

Harmonic maps from  $\mathbb{R}^2$  or one-connected domain  $\emptyset \subset \mathbb{R}^2$  into  $GL(m, \mathbb{C})$  and  $U(m)$  are treated. The GBDT version of the Bäcklund-Darboux transformation is applied to the case of the harmonic maps. A new general formula on the GBDT transformations of the Sym-Tafel immersions is derived. A class of the harmonic maps similar in certain ways to line-solitons is obtained explicitly and studied.

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*Keywords:* Harmonic map; Bäcklund-Darboux transformation; immersion; line soliton

## 1 Introduction

Harmonic maps are actively studied in differential geometry, mathematical physics and soliton theory, and the famous Bäcklund-Darboux transformation can be fruitfully used for this purpose. Since the original works of Bäcklund and Darboux various interesting versions of the Bäcklund-Darboux transformation have been introduced (see, for instance, [3, 4, 5, 6, 8, 9, 10,

23, 24, 25, 28] and references therein). Some of these versions have been successfully applied to the studies of harmonic maps.

In our paper we shall consider harmonic maps from  $\mathbb{R}^2$  or one-connected domain  $\mathcal{O} \subset \mathbb{R}^2$  into  $GL(m, \mathbb{C})$  and  $U(m)$ . Here  $GL(m, \mathbb{C})$  is the Lie group of  $m \times m$  invertible matrices with the complex valued entries, and  $U(m)$  is its subgroup of unitary matrices. Correspondingly the map  $u(x, y)$   $((x, y) \in \mathbb{R}^2)$  is called harmonic if it satisfies the Euler-Lagrange equation

$$\bar{\partial}((\partial u)u^{-1}) + \partial((\bar{\partial} u)u^{-1}) = 0 \quad \left( \partial = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \bar{\partial} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \right). \quad (1.1)$$

We apply a version of the Bäcklund-Darboux transformation (so called GBDT), developed in [12]-[17] and some other works of the author, to the case of the harmonic maps. A new general formula on the GBDT transformations of the Sym-Tafel immersions is derived. A class of the harmonic maps similar in certain ways to line-solitons is obtained explicitly and studied.

GBDT version of the Bäcklund-Darboux transformation for harmonic maps is constructed in Section 2. In particular, GBDT transformations of the Sym-Tafel immersions are given in Proposition 2.8. Explicit solutions are treated in Section 3.

## 2 GBDT version of the Bäcklund-Darboux transformation

Suppose  $u$  and its partial derivatives are continuously differentiable and  $u$  is a harmonic map into  $GL(m, \mathbb{C})$ . Then Euler-Lagrange equation (1.1) is equivalent [11] to the compatibility condition

$$\bar{\partial}G(x, y, \lambda) - \partial F(x, y, \lambda) + [G(x, y, \lambda), F(x, y, \lambda)] = 0, \quad ([G, F] := GF - FG) \quad (2.1)$$

for system

$$\partial w(x, y, \lambda) = G(x, y, \lambda)w(x, y, \lambda), \quad \bar{\partial}w(x, y, \lambda) = F(x, y, \lambda)w(x, y, \lambda), \quad (2.2)$$

where

$$G(x, y, \lambda) = -(\lambda - 1)^{-1}q(x, y), \quad F(x, y, \lambda) = -(\lambda + 1)^{-1}Q(x, y), \quad (2.3)$$

$$q(x, y) = (\partial u(x, y))u(x, y)^{-1}, \quad Q(x, y) = -(\bar{\partial} u(x, y))u(x, y)^{-1}. \quad (2.4)$$

Without loss of generality assume  $(0, 0) \in \emptyset$ . To construct GBDT fix an integer  $n > 0$  and five parameter matrices  $A_1, A_2, S(0, 0), \Pi_1(0, 0)$  and  $\Pi_2(0, 0)$ , where  $A_1, A_2, S$  are  $n \times n$  matrices, and  $\Pi_1, \Pi_2$  are  $n \times m$  matrices. We require also that  $\pm 1 \notin \sigma(A_k)$  ( $k = 1, 2, \sigma$  - spectrum) and the identity

$$A_1 S(0, 0) - S(0, 0) A_2 = \Pi_1(0, 0) \Pi_2(0, 0)^* \quad (2.5)$$

holds. Now introduce matrix functions  $\Pi_1(x, t)$  and  $\Pi_2(x, t)$  by their values at  $(x, y) = (0, 0)$  and linear differential equations

$$\partial \Pi_1 = (A_1 - I_n)^{-1} \Pi_1 q, \quad \bar{\partial} \Pi_1 = (A_1 + I_n)^{-1} \Pi_1 Q, \quad (2.6)$$

$$\partial \Pi_2^* = -q \Pi_2^* (A_2 - I_n)^{-1}, \quad \bar{\partial} \Pi_2^* = -Q \Pi_2^* (A_2 + I_n)^{-1}, \quad (2.7)$$

where  $I_n$  is the  $n \times n$  identity matrix. Similar to the case of differentiation in real valued arguments in [14, 17] the compatibility of both systems (2.6) and (2.7) follows from the compatibility condition (2.1).

Next, introduce  $m \times m$  matrix function  $S(x, y)$  via  $S(0, 0)$  and the partial derivatives

$$\partial S = -(A_1 - I_n)^{-1} \Pi_1 q \Pi_2^* (A_2 - I_n)^{-1}, \quad \bar{\partial} S = -(A_1 + I_n)^{-1} \Pi_1 Q \Pi_2^* (A_2 + I_n)^{-1}. \quad (2.8)$$

From (2.1) and (2.6)-(2.8) it follows that  $\bar{\partial} \partial S = \partial \bar{\partial} S$ , i.e.,  $S_{xy} = S_{yx}$ , where  $S_x = \frac{\partial S}{\partial x} = \partial S + \bar{\partial} S$ ,  $S_y = \frac{\partial S}{\partial y} = i(\partial S - \bar{\partial} S)$ . Thus the entries of  $S_x$  and  $S_y$  form potential fields and so equations (2.8) are compatible.

According to (2.6)-(2.8) we have  $\partial(A_1 S - S A_2) = \partial(\Pi_1 \Pi_2^*)$  and  $\bar{\partial}(A_1 S - S A_2) = \bar{\partial}(\Pi_1 \Pi_2^*)$ , which in view of (2.5) implies the identity

$$A_1 S(x, y) - S(x, y) A_2 = \Pi_1(x, y) \Pi_2(x, y)^*. \quad (2.9)$$

Assume now that  $S$  is invertible and consider well known in system theory transfer matrix function represented in the Lev Sakhnovich form [18]-[20]:

$$w_A(x, y, \lambda) = I_m - \Pi_2(x, y)^* S(x, y)^{-1} (A_1 - \lambda I_n)^{-1} \Pi_1(x, y). \quad (2.10)$$

The matrix function  $w_A$  is invertible [18]:

$$w_A(x, y, \lambda)^{-1} = I_m + \Pi_2(x, y)^* (A_2 - \lambda I_n)^{-1} S(x, y)^{-1} \Pi_1(x, y). \quad (2.11)$$

Formula (2.11) easily follows from the identity (2.9).

From Theorem 1 [14] (see also Appendix A [15]) it follows that  $w_A$  is a so called Darboux matrix function:

**Proposition 2.1** *Suppose  $m \times m$  matrix function  $u$  and its partial derivatives are continuously differentiable and relations (1.1), (2.4), and (2.5)-(2.8) are valid. Then in the points of invertibility of  $S$  we have*

$$\partial w_A(x, y, \lambda) = \tilde{G}(x, y, \lambda) w_A(x, y, \lambda) - w_A(x, y, \lambda) G(x, y, \lambda), \quad (2.12)$$

$$\bar{\partial} w_A(x, y, \lambda) = \tilde{F}(x, y, \lambda) w_A(x, y, \lambda) - w_A(x, y, \lambda) F(x, y, \lambda), \quad (2.13)$$

where  $w_A$  is given in (2.10),  $G$  and  $F$  are defined by (2.3),

$$\tilde{G}(x, y, \lambda) = -(\lambda - 1)^{-1} \tilde{q}(x, y), \quad \tilde{F}(x, y, \lambda) = -(\lambda + 1)^{-1} \tilde{Q}(x, y), \quad (2.14)$$

$$\tilde{q}(x, y) = w_A(x, y, 1) q(x, y) w_A(x, y, 1)^{-1}, \quad (2.15)$$

$$\tilde{Q}(x, y) = w_A(x, y, -1) Q(x, y) w_A(x, y, -1)^{-1}. \quad (2.16)$$

Suppose  $w$  is an  $m \times m$  invertible matrix function satisfying (2.2)-(2.4). By (2.2) and (2.3) the equalities

$$\partial w(x, y, 0) = q(x, y) w(x, y, 0), \quad \bar{\partial} w(x, y, 0) = -Q(x, y) w(x, y, 0) \quad (2.17)$$

hold. So in view of (2.4) one can normalize  $w$  so that

$$w(x, y, 0) = u(x, y), \quad (2.18)$$

where  $u$  satisfies the Euler-Lagrange equation (1.1). Normalized in this way matrix function  $w(x, y, \lambda)$  is called an extended (and corresponding to  $u$ ) solution of the Euler-Lagrange equation or extended frame.

**Theorem 2.2** *Suppose  $u$  satisfies the Euler-Lagrange equation (1.1), the conditions of Proposition 2.1 are fulfilled, and  $w$  is an extended corresponding to  $u$  solution. Then the matrix function*

$$\tilde{u}(x, y) := w_A(x, y, 0)u(x, y) \quad (2.19)$$

*also satisfies the Euler-Lagrange equation. Moreover the matrix function*

$$\tilde{w}(x, y, \lambda) := w_A(x, y, \lambda)w(x, y, \lambda). \quad (2.20)$$

*is an extended solution such that  $\tilde{w}(x, y, 0) = \tilde{u}(x, y)$ .*

**Proof.** According to formulas (2.2) and (2.20) and to Proposition 2.1 we have

$$\partial\tilde{w}(x, y, \lambda) = \tilde{G}(x, y, \lambda)\tilde{w}(x, y, \lambda), \quad \bar{\partial}\tilde{w}(x, y, \lambda) = \tilde{F}(x, y, \lambda)\tilde{w}(x, y, \lambda). \quad (2.21)$$

From (2.21) it follows that the compatibility condition

$$\bar{\partial}\tilde{G}(x, y, \lambda) - \partial\tilde{F}(x, y, \lambda) + [\tilde{G}(x, y, \lambda), \tilde{F}(x, y, \lambda)] = 0 \quad (2.22)$$

is fulfilled. Condition (2.22) can be rewritten in the form

$$(\lambda + 1)\bar{\partial}\tilde{q} - (\lambda - 1)\partial\tilde{Q} + [\tilde{Q}, \tilde{q}] = 0,$$

i.e., the coefficients of the polynomial in  $\lambda$  on the left-hand side of the last equation turn to zero. In other words we have

$$\bar{\partial}\tilde{q}(x, y) = \partial\tilde{Q}(x, y) = -\frac{1}{2}[\tilde{Q}(x, y), \tilde{q}(x, y)]. \quad (2.23)$$

From (2.18)-(2.20) it follows that  $\tilde{w}(x, y, 0) = \tilde{u}(x, y)$ . Therefore taking into account (2.14) and (2.21) we get

$$\partial\tilde{u}(x, y) = \tilde{q}(x, y)\tilde{u}(x, y), \quad \bar{\partial}\tilde{u}(x, y) = -\tilde{Q}(x, y)\tilde{u}(x, y). \quad (2.24)$$

Hence in view of (2.23) the matrix function  $\tilde{u}$  satisfies (1.1). Now we see that by (2.14), (2.21), and (2.24)  $\tilde{w}$  is an extended solution corresponding to  $\tilde{u}$ .

■

Theorem 2.2 presents a GBDT method to construct harmonic maps  $\tilde{u}$  into  $GL(m, \mathbb{C})$  and corresponding extended solutions. According to (2.10) the choice of the eigenvalues of  $A_1$  defines simple and multiple poles of the Darboux matrix  $w_A$ , and in this way our result is related to the interesting papers [2, 7, 27] on the pole data for the soliton solutions. Notice that we don't require parameter matrix  $A_1$  to be similar to diagonal (it may have an arbitrary Jordan structure).

Consider the case  $uu^* = u^*u = I_m$ , i.e.,  $u$  is a harmonic map into  $U(m)$ . Then we have  $u_x u^* + u u_x^* = u_y u^* + u u_y^* = 0$ . Therefore from (2.4) it follows that

$$q^* = \frac{1}{2} \left( u u_x^* + i u u_y^* \right) = -\frac{1}{2} \left( u_x u^* + i u_y u^* \right) = Q. \quad (2.25)$$

Put now  $A_1 = -A_2^* = A$  (i.e., assume  $A_1 = -A_2^*$ ). Then, taking into account (2.6), (2.7), and (2.25) we can assume  $\Pi_2 = \Pi_1$  and denote  $\Pi_1$  by  $\Pi$ . Further in this section we assume

$$A_1 = -A_2^* = A, \quad \Pi_1(x, t) = \Pi_2(x, t) = \Pi(x, t), \quad \text{and} \quad S(0, 0) = S(0, 0)^*. \quad (2.26)$$

Hence, using (2.8) and (2.25) we obtain  $(\partial S)^* = \bar{\partial} S$ , and so  $S_x = S_x^*$ ,  $S_y = S_y^*$ . The last equality in (2.26) now implies

$$S(x, y) = S(x, y)^*. \quad (2.27)$$

**Corollary 2.3** *Suppose the equality  $u^* = u^{-1}$ , the conditions of Theorem 2.2, and assumptions (2.26) are true. Then we have*

$$\tilde{u}(x, y)^* = \tilde{u}(x, y)^{-1}, \quad \tilde{q}(x, y)^* = \tilde{Q}(x, y). \quad (2.28)$$

*Proof.* Identity (2.9) and definition (2.10) now take the form

$$AS(x, y) + S(x, y)A^* = \Pi(x, y)\Pi(x, y)^*, \quad (2.29)$$

$$w_A(x, y, \lambda) = I_m - \Pi(x, y)^* S(x, y)^{-1} (A - \lambda I_n)^{-1} \Pi(x, y). \quad (2.30)$$

Formula (2.11) takes the form

$$w_A(x, y, \lambda)^{-1} = I_m - \Pi(x, y)^* (A^* + \lambda I_n)^{-1} S(x, y)^{-1} \Pi(x, y). \quad (2.31)$$

In view of (2.27), (2.30), and (2.31) we derive

$$w_A(x, y, \lambda)^{-1} = w_A(x, y, -\bar{\lambda})^* \quad (2.32)$$

In particular, we have  $w_A(x, y, 0)^{-1} = w_A(x, y, 0)^*$  and the first equality in (2.28) follows from the definition of  $\tilde{u}$ . Moreover, by (2.32) we have  $w_A(x, y, -1)^{-1} = w_A(x, y, 1)^*$  and so the second equality in (2.28) follows from (2.15), (2.16), and (2.25). ■

**Remark 2.4** *The uniton solutions have been introduced in the seminal paper [25]. Put*

$$A_1 = aI_n, \quad A_2 = bI_n, \quad \pi = (a - b)^{-1}\Pi_2^*S^{-1}\Pi_1. \quad (2.33)$$

*We shall derive some relations for  $\pi$  to compare with the limiting uniton case. By (2.9) and (2.33) we easily get*

$$\pi^2 = (a - b)^{-2}(\Pi_2^*S^{-1}A_1\Pi_1 - \Pi_2^*A_2S^{-1}\Pi_1) = \pi, \quad (2.34)$$

*i.e.,  $\pi(x, y)$  is a projector. From (2.7) and (2.8) it follows that*

$$\partial(\Pi_2^*S^{-1}) = -\tilde{q}\Pi_2^*S^{-1}(A_2 - I_n)^{-1}. \quad (2.35)$$

*Hence, taking into account (2.6) we have*

$$\partial(\Pi_2^*S^{-1}\Pi_1) = -\tilde{q}\Pi_2^*S^{-1}(A_2 - I_n)^{-1}\Pi_1 + \Pi_2^*S^{-1}(A_1 - I_n)^{-1}\Pi_1q. \quad (2.36)$$

*In view of (2.10), (2.11), (2.15), and (2.33) we obtain*

$$\tilde{q} = \left(I_m - \frac{a - b}{a - 1}\pi\right)q\left(I_m + \frac{a - b}{b - 1}\pi\right). \quad (2.37)$$

*Using (2.33), (2.34), and (2.37) rewrite (2.36) as*

$$\partial\pi = -(b - 1)^{-1}\frac{a - 1}{b - 1}\left(I_m - \frac{a - b}{a - 1}\pi\right)q\pi + (a - 1)^{-1}\pi q. \quad (2.38)$$

*When  $a = \bar{a}$ ,  $b = -a$ , we can assume  $\Pi_1 = \Pi_2$ ,  $S = S^*$ , i.e.,  $\pi = \pi^*$  is an orthogonal projector and*

$$\partial\pi = \frac{1 - a}{(a + 1)^2}\left(I_m - \frac{2a}{a - 1}\pi\right)q\pi + (a - 1)^{-1}\pi q. \quad (2.39)$$

The so called singular Bäcklund transformation that transforms unitons into unitons deals with the case  $a \rightarrow -1$ . In this case we have  $(a+1)^{-1} \rightarrow \infty$ , and so (2.39) implies  $(I_m - \pi)q\pi = 0$  (see [25] and formula (5.130) [6]). Similar transformations are possible for the equation

$$\bar{\partial}(\Pi_2^* S^{-1} \Pi_1) = -\tilde{Q} \Pi_2^* S^{-1} (A_2 + I_n)^{-1} \Pi_1 + \Pi_2^* S^{-1} (A_1 + I_n)^{-1} \Pi_1 Q.$$

It is of interest to consider harmonic maps into  $SU(m)$ . For that purpose introduce the notion of minimal realization. Any rational  $m \times m$  matrix function  $\varphi$  that tends to  $D$  at infinity can be presented in the form

$$\varphi(\lambda) = D - C(A - \lambda I_r)^{-1} B, \quad (2.40)$$

where  $D$  is an  $m \times m$  matrix,  $C$  is an  $m \times r$  matrix,  $B$  is an  $r \times m$  matrix, and  $A$  is an  $r \times r$  matrix,  $r \geq 0$ , and the case  $r = 0$  corresponds to  $\varphi \equiv D$ . This type representation is called a *realization* in system theory.

**Definition 2.5** *Realization (2.40) is called minimal if the order  $r$  of  $A$  is the minimal possible. This order is called the McMillan degree of  $\varphi$ .*

Realization remains minimal under small perturbations of  $A$ ,  $B$ ,  $C$ .

**Theorem 2.6** *Suppose the conditions of Corollary 2.3 are fulfilled,  $\sigma(A) \cap \sigma(-A^*) = \emptyset$ , and realization (2.30) is minimal for some  $(x_0, y_0) \in \emptyset$ . Then for each  $(x, y) \in \emptyset$  we have*

$$\det w_A(x, y, \lambda) = \prod_{k=1}^n \frac{\lambda + \overline{a_k}}{\lambda - a_k}, \quad (2.41)$$

where  $\{a_k\}$  are the eigenvalues of  $A$  taken with their algebraic multiplicity.

**Proof.** It is well known that as  $\sigma(A) \cap \sigma(-A^*) = \emptyset$  and (2.30) is a minimal realization of  $w_A(x_0, y_0, \lambda)$ , so  $\det w_A(x_0, y_0, \lambda)$  has poles in all the eigenvalues of  $A$  and only there. Moreover, if  $\det D \neq 0$ , then the McMillan degrees of  $\varphi$  and  $\varphi^{-1}$  coincide. Thus (2.31) is a minimal realization of  $w_A(x_0, y_0, \lambda)^{-1}$ . Hence  $\det w_A(x_0, y_0, \lambda)^{-1}$  has poles in all the eigenvalues of  $-A^*$  and only there. Therefore, if all the eigenvalues of  $A$  are simple we get

$$\det w_A(x_0, y_0, \lambda) = \prod_{k=1}^n \frac{\lambda + \overline{a_k}}{\lambda - a_k}. \quad (2.42)$$



If  $A$  has multiple eigenvalues we prove (2.42) by small perturbations of  $A$  (and corresponding perturbations of  $S$  so that the identity (2.29) preserves).

Finally, notice that  $\det w_A(x, y, \lambda) = \prod_{k=1}^r \frac{\lambda - \hat{a}_k}{\lambda - a_k}$  for each  $(x, y)$  with possible arbitrariness in the choice of the set of eigenvalues  $a_k$  of  $A$  and  $\hat{a}_k$  of  $-A^*$ . Taking into account that  $w_A$  is also continuous we derive (2.41) from (2.42).  $\blacksquare$

The next Corollary is immediate.

**Corollary 2.7** *Suppose the conditions of Theorem 2.6 are fulfilled,  $\det(iA) \in \mathbb{R}$ , and  $u \in SU(m)$ . Then we have  $\tilde{u} \in SU(m)$ .*

In the framework of his theory of "soliton surfaces" A. Sym associates with integrable nonlinear systems the corresponding  $\lambda$ -families of immersions

$$R(x, y, \lambda) = w(x, y, \lambda)^{-1} \frac{\partial}{\partial \lambda} w(x, y, \lambda), \quad (2.43)$$

where  $w$  are extended solutions [22]. On the other hand, our version of the Darboux matrix, that can be used in numerous important cases, including harmonic maps treated in this section, admits representation (2.10), where dependence on  $\lambda$  is restricted to the resolvent  $(A_1 - \lambda I_n)^{-1}$ . Thus  $w_A$  is easily differentiated in  $\lambda$ . The next proposition expresses immersions (2.43) generated by GBDT in terms of  $\Pi_1$ ,  $\Pi_2$ , and  $S$ .

**Proposition 2.8** *Suppose extended solution  $\tilde{w}$  is given by equality (2.20), where Darboux matrix  $w_A$  admits representation (2.10) and identity (2.9) holds. Then immersion  $\tilde{R} := \tilde{w}^{-1} \frac{\partial}{\partial \lambda} \tilde{w}$  admits representation*

$$\tilde{R} = R - w^{-1} \Pi_2^* (A_2 - \lambda I_n)^{-1} S^{-1} (A_1 - \lambda I_n)^{-1} \Pi_1 w. \quad (2.44)$$

*Proof.* From (2.20) it easily follows that

$$\tilde{R} = R + w^{-1} w_A^{-1} \left( \frac{\partial}{\partial \lambda} w_A \right) w. \quad (2.45)$$

By (2.10) we have

$$\frac{\partial}{\partial \lambda} w_A(x, y, \lambda) = -\Pi_2(x, y)^* S(x, y)^{-1} (A_1 - \lambda I_n)^{-2} \Pi_1(x, y). \quad (2.46)$$

Taking into account (2.9), (2.11), and (2.46) one gets

$$\begin{aligned}
w_A^{-1} \frac{\partial}{\partial \lambda} w_A &= \frac{\partial}{\partial \lambda} w_A - \Pi_2^* (A_2 - \lambda I_n)^{-1} S^{-1} \Pi_1 \Pi_2^* S^{-1} (A_1 - \lambda I_n)^{-2} \Pi_1 \\
&= \frac{\partial}{\partial \lambda} w_A - \Pi_2^* (A_2 - \lambda I_n)^{-1} S^{-1} \left( (A_1 - \lambda I_n) S - S (A_2 - \lambda I_n) \right) S^{-1} (A_1 - \lambda I_n)^{-2} \Pi_1 \\
&= -\Pi_2^* (A_2 - \lambda I_n)^{-1} S^{-1} (A_1 - \lambda I_n)^{-1} \Pi_1.
\end{aligned} \tag{2.47}$$

Substitute now (2.47) into (2.45) to get (2.44). ■

Proposition 2.8 can be used to construct conformal CMC immersions.

### 3 Explicit solutions

For the simple seed solutions (see, for instance, solutions given by formulas (3.1) and (3.32)) equations (2.6) and (2.29) can be usually solved explicitly, so that equalities (2.19) and (2.30) provide us with the explicit expressions for harmonic maps. In this section we consider several examples in greater detail. First similar to [6] we put  $m = 2$  and take the most simple seed solution of (1.1):

$$u(x, y) = e^{(\tau \bar{z} - \bar{\tau} z)j} \in U(2), \quad z = x + iy, \quad \tau \neq 0, \quad j = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{3.1}$$

Transformations of this seed solution are treated in Examples 3.1-3.3. Example 3.4 deals with the arbitrary  $m$  and seed solution of the form (3.32). It is of interest in Examples 3.1, 3.3 that asymptotics of the constructed maps differs in one particular direction and non-diagonal entries tend to zero, except, possibly, in this direction. (See relations (3.9), (3.11), (3.28), and (3.31).)

For  $u$  of the form (3.1) by (2.4) we obtain

$$q(x, y) = -\bar{\tau}j, \quad Q(x, y) = -\tau j. \tag{3.2}$$

Partition the matrix function  $\Pi$  into columns

$$\Pi(x, y) = \begin{bmatrix} \Phi_1(x, y) & \Phi_2(x, y) \end{bmatrix}, \quad \Pi(0, 0) = \begin{bmatrix} f_1 & f_2 \end{bmatrix}. \tag{3.3}$$

Then according to (2.6), (2.26), and (3.2) we have

$$\Phi_1(x, y) = \exp \left( -\tau \bar{z}(A + I_n)^{-1} - \bar{\tau} z(A - I_n)^{-1} \right) f_1, \quad (3.4)$$

$$\Phi_2(x, y) = \exp \left( \tau \bar{z}(A + I_n)^{-1} + \bar{\tau} z(A - I_n)^{-1} \right) f_2. \quad (3.5)$$

Consider now the case  $n = 1$ ,  $A = a$  ( $a \neq \pm 1$ ,  $a \neq -\bar{a}$ ),  $f_1, f_2 \neq 0$ . Recall that the one-soliton solution treated in [6] (formulas (5.101), (5.102)) had the following behavior on the lines  $z = x + iy = \mu t$  ( $\mu \in \mathbb{C}$ ,  $\mu \neq 0$ ,  $-\infty < t < \infty$ ):

$$\tilde{u} = gu, \quad g(x, y) = c_{\pm} e^{|kt|} (I_2 + o(1)) \quad \text{for } t \rightarrow \pm\infty, \quad c_{\pm} \in \mathbb{R}. \quad (3.6)$$

Our next example proves quite different.

**Example 3.1** *The GBDT transformation  $\tilde{u}$  of the seed solution is given by formula (2.19):*

$$\tilde{u}(x, y) = w_A(x, y, 0)u(x, y),$$

where  $w_A$  is defined in (2.30). Let us study  $\tilde{u}$  on the lines  $z = \mu t$ . For  $\Pi$  in the right hand side of (2.30), by (3.3)-(3.5) we get

$$\Pi(x, y) = \begin{bmatrix} e^{-bt} f_1 & e^{bt} f_2 \end{bmatrix}, \quad b := (a - 1)^{-1} \bar{\tau} \mu + (a + 1)^{-1} \tau \bar{\mu}. \quad (3.7)$$

Hence from (2.29) and (3.7) it follows that

$$S(x, y) = (a + \bar{a})^{-1} \Pi(x, y) \Pi(x, y)^* = (a + \bar{a})^{-1} \left( e^{-(b+\bar{b})t} |f_1|^2 + e^{(b+\bar{b})t} |f_2|^2 \right). \quad (3.8)$$

Without loss of generality we can assume  $b + \bar{b} \geq 0$ . In view of (2.30), (3.7), and (3.8) for  $b + \bar{b} > 0$  we have

$$\lim_{t \rightarrow \infty} w_A(x, y, 0) = \begin{bmatrix} 1 & 0 \\ 0 & -\bar{a}/a \end{bmatrix}, \quad \lim_{t \rightarrow -\infty} w_A(x, y, 0) = \begin{bmatrix} -\bar{a}/a & 0 \\ 0 & 1 \end{bmatrix}. \quad (3.9)$$

The asymptotics differs on the line  $\Gamma_0 = \{z : z = \mu t\}$ , where  $b + \bar{b} = 0$ , i.e.,

$$\mu = \frac{i\tau c}{(\bar{a} - 1)(a + 1)}, \quad c = \bar{c} \neq 0. \quad (3.10)$$

Namely, on  $\Gamma_0$  we have

$$w_A(x, y, 0) = \frac{1}{a} \begin{bmatrix} \alpha & \beta(t) \\ \frac{\alpha}{\beta(t)} & -\bar{\alpha} \end{bmatrix}, \quad \alpha = \frac{a|f_2|^2 - \bar{a}|f_1|^2}{|f_1|^2 + |f_2|^2}, \quad (3.11)$$

$$\beta(t) = -\frac{(a + \bar{a})\bar{f}_1 f_2}{|f_1|^2 + |f_2|^2} e^{2bt} \quad (\bar{b} = -b). \quad (3.12)$$

Similar to the "line solitons" studied, in particular, for KP (see, for instance, [1, 21, 26]) the asymptotics of our solution on  $\Gamma_0$  essentially differs from asymptotics along other lines. Notice also that only on  $\Gamma_0$  the non-diagonal entries of  $\tilde{u}$  do not decay to zero.

**Example 3.2** *This example deals with the case  $n = 2$ ,  $A = \text{diag}\{a_1, a_2\}$ , where  $\text{diag}$  means diagonal matrix,  $a_k \neq \pm 1$ , and  $\sigma(A) \cap \sigma(-A^*) = \emptyset$ . By (3.3)-(3.5) on a line  $z = \mu t$  we have*

$$\Pi(x, y) = \begin{bmatrix} e^{-Bt} f_1 & e^{Bt} f_2 \end{bmatrix}, \quad B = \text{diag}\{b_1, b_2\}, \quad (3.13)$$

$$b_k := (a_k - 1)^{-1} \bar{\tau} \mu + (a_k + 1)^{-1} \tau \bar{\mu} \quad (k = 1, 2).$$

Supposing  $b_1 + \bar{b}_1 \neq 0$  we choose the sign of  $\mu$  so that  $b_1 + \bar{b}_1 > 0$ . Assume that  $b_2 + \bar{b}_2 > 0$  too (the case  $b_2 + \bar{b}_2 < 0$  can be treated similar). Let the entries  $f_{12}$  and  $f_{22}$  of  $f_2$  be nonzero and let  $a_1 \neq a_2$ . Then in view of (2.29) and (3.13) we easily get

$$\Pi(x, y) = \begin{bmatrix} 0 & e^{b_1 t} f_{12} \\ 0 & e^{b_2 t} f_{22} \end{bmatrix} + o(1), \quad t \rightarrow \infty, \quad (3.14)$$

$$S(x, y)^{-1} = \frac{1}{\det S(x, y)} \left( \begin{bmatrix} \frac{|f_{22}|^2}{a_2 + \bar{a}_2} e^{(b_2 + \bar{b}_2)t} & -\frac{f_{12} \bar{f}_{22}}{a_1 + \bar{a}_2} e^{(b_1 + \bar{b}_2)t} \\ -\frac{f_{22} \bar{f}_{12}}{a_2 + \bar{a}_1} e^{(b_2 + \bar{b}_1)t} & \frac{|f_{12}|^2}{a_1 + \bar{a}_1} e^{(b_1 + \bar{b}_1)t} \end{bmatrix} + o(1) \right), \quad (3.15)$$

$$\det S(x, y) = (1 + o(1)) \frac{|f_{12} f_{22}|^2 (|a_1|^2 + |a_2|^2 - a_1 \bar{a}_2 - a_2 \bar{a}_1)}{(a_1 + \bar{a}_1)(a_2 + \bar{a}_2) |a_1 + \bar{a}_2|^2} e^{(b_1 + \bar{b}_1 + b_2 + \bar{b}_2)t}. \quad (3.16)$$

After some calculations the asymptotics of  $w_A(x, y, 0) = I_2 - \Pi^* S^{-1} A^{-1} \Pi$  follows from (3.14)-(3.16):

$$\lim_{t \rightarrow \infty} w_A(x, y, 0) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{\bar{a}_1 \bar{a}_2}{a_1 a_2} \end{bmatrix}. \quad (3.17)$$

If in the last example, where  $A$  is diagonal, we put  $a_1 = a_2 = a$  and  $\det \Pi(0, 0) \neq 0$ , we easily get a trivial answer  $w_A(x, y, 0) = -(\bar{a}/a)I_2$ . The case, where  $A$  is a Jordan box, is far more interesting.

**Example 3.3** Suppose now that

$$n = 2, \quad A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}, \quad a \neq \pm 1, \quad a \neq -\bar{a}. \quad (3.18)$$

It follows that

$$\tau \bar{\mu}(A + I_2)^{-1} + \bar{\tau} \mu(A - I_2)^{-1} = bI_2 + cR, \quad R = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad (3.19)$$

where  $b$  is given by the second relation in (3.7) and

$$c = -((a - 1)^{-2} \bar{\tau} \mu + (a + 1)^{-2} \tau \bar{\mu}). \quad (3.20)$$

According to (3.3)-(3.5) and (3.19) on a line  $z = x + iy = \mu t$  we have

$$\Pi(x, y) = [e^{-bt}(I_2 - ctR)f_1 \quad e^{bt}(I_2 + ctR)f_2]. \quad (3.21)$$

In view of (3.18) identity (2.29) takes the form

$$(a + \bar{a})S + \begin{bmatrix} s_{21} + s_{12} & s_{22} \\ s_{22} & 0 \end{bmatrix} = \Pi \Pi^*, \quad (3.22)$$

where  $s_{kj}$  are the entries of  $S$ . From (3.21) and (3.22) it is immediate that

$$s_{22}(x, y) = (a + \bar{a})^{-1} \left( e^{-(b+\bar{b})t} |f_{21}|^2 + e^{(b+\bar{b})t} |f_{22}|^2 \right), \quad (3.23)$$

$$s_{12}(x, y) = \overline{s_{21}(x, y)} = (a + \bar{a})^{-1}$$

$$\times \left( e^{-(b+\bar{b})t} (f_{11} - ct f_{21}) \overline{f_{21}} + e^{(b+\bar{b})t} (f_{12} + ct f_{22}) \overline{f_{22}} - s_{22}(x, y) \right), \quad (3.24)$$

$$s_{11}(x, y) = (a + \bar{a})^{-1}$$

$$\times \left( e^{-(b+\bar{b})t} |f_{11} - ct f_{21}|^2 + e^{(b+\bar{b})t} |f_{12} + ct f_{22}|^2 - s_{12}(x, y) - s_{21}(x, y) \right). \quad (3.25)$$

Similar to Example 3.1 assume  $b + \bar{b} \geq 0$ . Let  $f_{21} \neq 0$  and  $f_{22} \neq 0$  for simplicity. First we shall treat the lines, where  $b + \bar{b} > 0$ . Taking into account (3.23)-(3.25) by standard calculations we derive

$$\det S(x, y) = (1 + o(1))(a + \bar{a})^{-4} e^{2(b+\bar{b})t} |f_{22}|^4. \quad (3.26)$$

One can easily see that

$$\Pi^* S^{-1} A^{-1} \Pi = a^{-2} (\det S)^{-1} \Pi^* \begin{bmatrix} s_{22} & -s_{12} \\ -s_{21} & s_{11} \end{bmatrix} \begin{bmatrix} a & -1 \\ 0 & a \end{bmatrix} \Pi. \quad (3.27)$$

From (3.23)-(3.27) it follows that

$$\Pi^* S^{-1} A^{-1} \Pi \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & a^{-2}(a^2 - \bar{a}^2) \end{bmatrix} \quad \text{when } t \rightarrow \infty, \quad i.e.,$$

$$\lim_{t \rightarrow \infty} w_A(x, y, 0) = \begin{bmatrix} 1 & 0 \\ 0 & (a^{-1} \bar{a})^2 \end{bmatrix} \quad \text{for } b + \bar{b} > 0. \quad (3.28)$$

As in Example 3.1, the asymptotics of  $w_A(x, y, 0)$  differs on the line  $\Gamma_0$ , where  $b + \bar{b} = 0$ . For this line by (3.23)-(3.25) we obtain

$$\det S = 4(a + \bar{a})^{-2} |cf_{21} f_{22}|^2 t^2 + k_1 t + k_0. \quad (3.29)$$

Taking also into account (3.27) we have

$$\Pi^* S^{-1} A^{-1} \Pi = 4a^{-1} (a + \bar{a})^{-1} |cf_{21} f_{22}|^2 (\det S)^{-1} \begin{bmatrix} t^2 + O(t) & O(t) \\ O(t) & t^2 + O(t) \end{bmatrix}. \quad (3.30)$$

Formulas (3.29) and (3.30) imply

$$\lim_{t \rightarrow \infty} w_A(x, y, 0) = -(\bar{a}/a)I_2 \quad \text{for } b + \bar{b} = 0. \quad (3.31)$$

Compare equalities (3.28) and (3.31). We would like to mention here that the extended solution  $\tilde{w} = w_A u$  has the pole of order two at  $\lambda = a$ :

$$\begin{aligned} \tilde{w}(t, \lambda) &= (a - \lambda)^{-2} \\ &\times (\det S(t))^{-1} \Pi(t)^* \begin{bmatrix} s_{22}(t) \\ -s_{21}(t) \end{bmatrix} \begin{bmatrix} e^{-bt} f_{21} & e^{bt} f_{22} \end{bmatrix} + O((a - \lambda)^{-1}). \end{aligned}$$

Our next example deals with an arbitrary  $m$  and a seed solution somewhat more general than the one given in (3.1). Namely we put

$$u(x, y) = e^{\bar{z}D - zD^*} \in U(m), \quad D = \text{diag}\{d_1, d_2, \dots, d_m\}. \quad (3.32)$$

It follows that

$$q(x, y) = -D^*, \quad Q(x, y) = -D. \quad (3.33)$$

**Example 3.4** As in Example 3.1, suppose  $n = 1$ ,  $A = a$  ( $a \neq \pm 1$ ,  $a \neq -\bar{a}$ ),  $f_k \neq 0$ . Here  $1 \leq k \leq m$ . Then, on a line  $z = \mu t$  taking into account (2.6) and (2.29) we obtain

$$\Pi(x, y) = \begin{bmatrix} e^{-b_1 t} f_1 & e^{-b_2 t} f_2 & \dots \end{bmatrix}, \quad b_k := (a - 1)^{-1} \bar{d}_k \mu + (a + 1)^{-1} d_k \bar{\mu}, \quad (3.34)$$

$$S(x, y) = (a + \bar{a})^{-1} \sum_{k=1}^m |f_k|^2 e^{-(b_k + \bar{b}_k)t} \quad (3.35)$$

In the generic situation there exists only one natural number  $k_+$  and only one natural number  $k_-$  such that

$$\Re b_{k_+} = \min_{1 \leq k \leq m} \Re b_k, \quad \Re b_{k_-} = \max_{1 \leq k \leq m} \Re b_k. \quad (3.36)$$

Then by (3.34) and (3.36) we have

$$\begin{aligned} \lim_{t \rightarrow \infty} w_A(x, y, 0) &= \text{diag}\{1, \dots, 1, -\bar{a}/a, 1, \dots\}, \text{ where } -\bar{a}/a \text{ is the } k_+ \text{th entry,} \\ \lim_{t \rightarrow -\infty} w_A(x, y, 0) &= \text{diag}\{1, \dots, 1, -\bar{a}/a, 1, \dots\}, \text{ where } -\bar{a}/a \text{ is the } k_- \text{th entry.} \end{aligned}$$

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